On the Passage of Waves through Fine Slits in Thin Opaque Screens.

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In a former paper* I gave solutions applicable to the passage of light through very narrow slits in infinitely thin perfectly opaque screens, for the two principal cases where the polarisation is either parallel or perpendicular to the length of the slit. It appeared that if the width ($2b$) of the slit is very small in comparison with the wave-length (λ), there is a much more free passage when the electric vector is perpendicular to the slit than when it is parallel to the slit, so that unpolarised light incident upon the screen will, after passage, appear polarised in the former manner. This conclusion is in accordance with the observations of Fizeau† upon the very narrowest slits. Fizeau found, however, that somewhat wider slits (scratches upon silvered glass) gave the opposite polarisation; and I have long wished to extend the calculations to slits of width comparable with λ. The subject has also a practical interest in connection with observations upon the Zeeman effect.‡

The analysis appropriate to problems of this sort would appear to be by use of elliptic co-ordinates; but I have not seen my way to a solution on these lines, which would, in any case, be rather complicated. In default of such a solution, I have fallen back upon the approximate methods of my former paper. Apart from the intended application, some of the problems which present themselves have an interest of their own. It will be convenient to repeat the general argument almost in the words formerly employed.

Plane waves of simple type impinge upon a parallel screen. The screen is supposed to be infinitely thin and to be perforated by some kind of aperture. Ultimately, one or both dimensions of the aperture will be regarded as small, or, at any rate, as not large, in comparison with the wave-length (λ); and the investigation commences by adapting to the present purpose known solutions concerning the flow of incompressible fluids.

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* "On the Passage of Waves through Apertures in Plane Screens and Allied Problems," 'Phil. Mag.,' 1897, vol. 43, p. 259; 'Scientific Papers,' vol. 4, p. 283.

† 'Annales de Chimie,' 1861, vol. 63, p. 385; Mascart's 'Traité d'Optique,' § 645. See also 'Phil. Mag.,' 1907, vol. 14, p. 350; 'Scientific Papers,' vol. 5, p. 417.

‡ Zeeman, 'Amsterdam Proceedings,' October, 1912.

The functions that we require may be regarded as velocity-potentials ϕ, satisfying

$$d^2\phi/dt^2 = \mathrm{V}\nabla^2\phi, \tag{1}$$

where

$$\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2,$$

and V is the velocity of propagation. If we assume that the vibration is everywhere proportional to e^{int}, (1) becomes

$$(\nabla^2 + k^2)\,\phi = 0, \tag{2}$$

where

$$k = n/\mathrm{V} = 2\pi/\lambda. \tag{3}$$

It will conduce to brevity if we suppress the factor e^{int}. On this understanding the equation of waves travelling parallel to x in the positive direction, and accordingly incident upon the negative side of the screen situated at $x = 0$, is

$$\phi = e^{-ikx}. \tag{4}$$

When the solution is complete, the factor e^{int} is to be restored, and the imaginary part of the solution is to be rejected. The realised expression for the incident waves will therefore be

$$\phi = \cos(nt - kx). \tag{5}$$

There are two cases to be considered corresponding to two alternative boundary conditions. In the first (i) $d\phi/dn = 0$ over the unperforated part of the screen, and in the second (ii) $\phi = 0$. In case (i) dn is drawn outwards normally, and if we take the axis of z parallel to the length of the slit, ϕ will represent the magnetic component parallel to z, usually denoted by c, so that this case refers to vibrations for which the electric vector is perpendicular to the slit. In the second case (ii) ϕ is to be identified with the component parallel to z of the electric vector R, which vanishes upon the walls, regarded as perfectly conducting. We proceed with the further consideration of case (i).

If the screen be complete, the reflected waves under condition (i) have the expression $\phi = e^{ikx}$. Let us divide the actual solution into two parts, χ and ψ; the first, the solution which would obtain were the screen complete; the second, the alteration required to take account of the aperture; and let us distinguish by the suffixes m and p the values applicable upon the negative (*minus*), and upon the positive side of the screen. In the present case we have

$$\chi_m = e^{-ikx} + e^{ikx}, \qquad \chi_p = 0. \tag{6}$$

This χ-solution makes $d\chi_m/dn = 0$, $d\chi_p/dn = 0$ over the whole plane $x = 0$, and over the same plane $\chi_m = 2$, $\chi_p = 0$.

For the supplementary solution, distinguished in like manner upon the two sides, we have

$$\psi_m = \iint \Psi_m \frac{e^{-ikr}}{r} d\mathrm{S}, \qquad \psi_p = \iint \Psi_p \frac{e^{-ikr}}{r} d\mathrm{S}, \tag{7}$$

where r denotes the distance of the point at which ψ is to be estimated from the element $d\mathrm{S}$ of the aperture, and the integration is extended over the whole of the area of aperture. Whatever functions of position Ψ_m, Ψ_p may be, these values on the two sides satisfy (2), and (as is evident from symmetry) they make $d\psi_m/dn$, $d\psi_p/dn$ vanish over the wall, viz., the unperforated part of the screen, so that the required condition over the wall for the complete solution is already satisfied. It remains to consider the further conditions that ϕ and $d\phi/dx$ shall be continuous across the *aperture*. These conditions require that on the aperture

$$2 + \psi_m = \psi_p, \qquad d\psi_m/dx = d\psi_p/dx. \tag{8}^*$$

The second is satisfied if $\Psi_p = -\Psi_m$; so that

$$\psi_m = \iint \Psi_m \frac{e^{-ikr}}{r} d\mathrm{S}, \qquad \psi_p = -\iint \Psi_m \frac{e^{-ikr}}{r} d\mathrm{S}, \tag{9}$$

making the values of ψ_m, ψ_p equal and opposite at all corresponding points, viz., points which are images of one another in the plane $x = 0$. In order further to satisfy the first condition, it suffices that over the area of aperture

$$\psi_m = -1, \qquad \psi_p = 1, \tag{10}$$

and the remainder of the problem consists in so determining Ψ_m that this shall be the case.

It should be remarked that Ψ in (9) is closely connected with the normal velocity at $d\mathrm{S}$. In general,

$$\frac{d\psi}{dx} = \iint \Psi \frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right) d\mathrm{S}. \tag{11}$$

At a point (x) infinitely close to the surface, only the neighbouring elements contribute to the integral, and the factor e^{-ikr} may be omitted. Thus

$$\frac{d\psi}{dx} = -\iint \Psi \frac{x}{r^2} d\mathrm{S} = -2\pi x \int_x^\infty \Psi \frac{r\,dr}{r^3} = -2\pi\Psi; \quad \text{or} \quad \Psi = -\frac{1}{2\pi}\frac{d\psi}{dn}, \tag{12}$$

$d\psi/dn$ being the normal velocity at the point of the surface in question.

In the original paper these results were applied to an aperture, especially of elliptical form, whose dimensions are small in comparison with λ. For our present purpose we may pass this over and proceed at once to consider the case where the aperture is an infinitely long slit with parallel edges, whose width is small, or at the most comparable with λ.

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* The use of dx implies that the variation is in a fixed direction, while dn may be supposed to be drawn outwards from the screen in both cases.

The velocity-potential of a point-source, viz., e^{-ikr}/r, is now to be replaced by that of a linear source, and this, in general, is much more complicated. If we denote it by $D(kr)$, r being the distance from the line of the point where the potential is required, the expressions are*

$$D(kr) = -\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\left\{1-\frac{1^2}{1.8ikr}+\frac{1^2.3^2}{1.2.(8ikr)^2}-\ldots\right\}$$

$$= \left(\gamma+\log\frac{ikr}{2}\right)\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}-\ldots\right\}$$

$$+\frac{k^2r^2}{2^2}S_1-\frac{k^4r^4}{2^2.4^2}S_2+\frac{k^6r^6}{2^2.4^2.6^2}S_3-\ldots, \qquad (13)$$

where γ is Euler's constant (0·577215), and

$$S_m = 1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+1/m. \qquad (14)$$

Of these the first is "semi-convergent" and is applicable when kr is large; the second is fully convergent and gives the form of the function when kr is moderate. The function D may be regarded as being derived from e^{-ikr}/r by integration over an infinitely long and infinitely narrow strip of the surface S.

As the present problem is only a particular case, equations (6) and (10) remain valid, while (9) may be written in the form

$$\psi_m = \int\Psi_m D(kr)\,dy, \qquad \psi_p = -\int\Psi_m D(kr)\,dy, \qquad (15)$$

the integrations extending over the width of the slit from $y = -b$ to $y = +b$. It remains to determine Ψ_m, so that on the aperture $\psi_m = -1$, $\psi_p = +1$.

At a sufficient distance from the slit, supposed for the moment to be very narrow, $D(kr)$ may be removed from under the integral sign and also be replaced by its limiting form given in (13). Thus

$$\psi_m = -\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\int\Psi_m dy. \qquad (16)$$

If the slit be not very narrow, the partial waves arising at different parts of the width will arrive in various phases, of which due account must be taken. The disturbance is no longer circularly symmetrical as in (16). But if, as is usual in observations with the microscope, we restrict ourselves to the direction of original propagation, equality of phase obtains, and (16) remains applicable even in the case of a wide slit. It only remains to determine Ψ_m as a function of y, so that for all points upon the aperture

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* See 'Theory of Sound,' § 341.

$$\int_{-b}^{+b} \Psi_m \mathrm{D}(kr)\, dy = -1, \tag{17}$$

where, since kr is supposed moderate throughout, the second form in (13) may be employed.

Before proceeding further it may be well to exhibit the solution, as formerly given, for the case of a very narrow slit. Interpreting ϕ as the velocity-potential of aerial vibrations and having regard to the known solution for the flow of incompressible fluid through a slit in an infinite plane wall, we may infer that Ψ_m will be of the form $\mathrm{A}\,(b^2-y^2)^{-\frac{1}{2}}$, where A is some constant. Thus (17) becomes

$$\mathrm{A}\left[(\gamma+\log \tfrac{1}{2} ik)\,\pi + \int_{-b}^{+b} \frac{\log r \,.\, dy}{\sqrt{(b^2-y^2)}}\right] = -1. \tag{18}$$

In this equation the first part is obviously independent of the position of the point chosen, and if the form of Ψ_m has been rightly taken the second integral must also be independent of it. If its co-ordinate be η, lying between $\pm b$,

$$\int_{-b}^{+b} \frac{\log r \,.\, dy}{\sqrt{(b^2-y^2)}} = \int_{-b}^{\eta} \frac{\log(\eta-y)\, dy}{\sqrt{(b^2-y^2)}} + \int_{\eta}^{b} \frac{\log(y-\eta)\, dy}{\sqrt{(b^2-y^2)}} \tag{19}$$

must be independent of η. To this we shall presently return; but merely to determine A in (18) it suffices to consider the particular case of $\eta = 0$. Here

$$\int_{-b}^{+b} \frac{\log r \,.\, dy}{\sqrt{(b^2-y^2)}} = 2\int_{0}^{b} \frac{\log y \,.\, dy}{\sqrt{(b^2-y^2)}} = 2\int_{0}^{\frac{1}{2}\pi} \log(b\cos\theta)\, d\theta = \pi \log(\tfrac{1}{2}b).$$

Thus $\quad \mathrm{A}\,(\gamma+\log \tfrac{1}{4} ikb)\,\pi = -1, \quad$ and $\quad \int_{-b}^{+b} \Psi_m dy = \pi \mathrm{A}\,;$

so that (16) becomes

$$\psi_m = \frac{e^{-ikr}}{\gamma+\log(\frac{1}{4}ikb)} \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \tag{20}$$

From this, ψ_p is derived by simply prefixing a negative sign.

The realised solution is obtained from (20) by omitting the imaginary part after introduction of the suppressed factor e^{int}. If the imaginary part of $\log(\frac{1}{4}\, ikb)$ be neglected, the result is

$$\psi_m = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \frac{\cos(nt-kr-\frac{1}{4}\pi)}{\gamma+\log(\frac{1}{4}kb)}, \tag{21}$$

corresponding to $\quad \chi_m = 2\cos nt \cos kx. \tag{22}$

Perhaps the most remarkable feature of the solution is the very limited dependence of the transmitted vibration on the *width* ($2b$) of the aperture.

We will now verify that (19) is independent of the special value of η. Writing $y = b\cos\theta$, $\eta = b\cos\alpha$, we have

$$\int_{-b}^{+b}\frac{\log r\,.\,dy}{\sqrt{(b^2-y^2)}} = \int_0^{\pi}\log(\tfrac{1}{2}b)\,d\theta + \int_0^{\alpha}\log 2(\cos\theta-\cos\alpha)\,d\theta$$

$$+\int_{\alpha}^{\pi}\log 2(\cos\alpha-\cos\theta)\,d\theta = \pi\log(\tfrac{1}{2}b)$$

$$+\int_0^{\pi}\log\left\{2\sin\frac{\alpha+\theta}{2}\right\}d\theta + \int_0^{\alpha}\log\left\{2\sin\frac{\alpha-\theta}{2}\right\}d\theta + \int_{\alpha}^{\pi}\log\left\{2\sin\frac{\theta-\alpha}{2}\right\}d\theta$$

$$= \pi\log\tfrac{1}{2}b + 2\int_{\frac{1}{2}\alpha}^{\frac{1}{2}\pi+\frac{1}{2}\alpha}\log(2\sin\phi)\,d\phi + \int_0^{\frac{1}{2}\alpha}\log(2\sin\phi)\,d\phi + 2\int_0^{\frac{1}{2}\pi-\frac{1}{2}\alpha}\log(2\sin\phi)\,d\phi$$

$$= \pi\log\tfrac{1}{2}b + 2\int_0^{\frac{1}{2}\pi}\log(2\sin\phi)\,d\phi + 2\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\frac{1}{2}\alpha}\log(2\sin\phi)\,d\phi + 2\int_0^{\frac{1}{2}\pi-\frac{1}{2}\alpha}\log(2\sin\phi)\,d\phi$$

$$= \pi\log\tfrac{1}{2}b + 4\int_0^{\frac{1}{2}\pi}\log(2\sin\phi)\,d\phi,$$

as we see by changing ϕ into $\pi-\phi$ in the second integral. Since α has disappeared, the original integral is independent of η. In fact,*

$$\int_0^{\frac{1}{2}\pi}\log(2\sin\phi)\,d\phi = 0,$$

and we have

$$\int_{-b}^{+b}\frac{\log r\,.\,dy}{\sqrt{(b^2-y^2)}} = \pi\log\tfrac{1}{2}b, \tag{23}$$

as in the particular case of $\eta = 0$.

The required condition (17) can thus be satisfied by the proposed form of Ψ, provided that kb be small enough. When kb is greater, the resulting value of ψ in (15) will no longer be constant over the aperture, but we may find what the actual value is as a function of η by carrying out the integration with inclusion of more terms in the series representing D. As a preliminary, it will be convenient to discuss certain definite integrals which present themselves. The first of the series, which has already occurred, we will call h_0, so that

$$h_0 = \int_0^{\frac{1}{2}\pi}\log(2\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi}\log(2\cos\theta)\,d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\log(2\sin 2\theta)\,d\theta$$

$$= \tfrac{1}{4}\int_0^{\pi}\log(2\sin\phi)\,d\phi = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\log(2\sin\phi)\,d\phi = \tfrac{1}{2}h_0.$$

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* See below.

Accordingly, $h_0 = 0$. More generally we set, n being any even integer,

$$h_n = \int_0^{\frac{1}{2}\pi} \sin^n \theta \log(2 \sin \theta)\, d\theta, \tag{24}$$

or, on integration by parts,

$$\begin{aligned} h_n &= \int_0^{\frac{1}{2}\pi} \cos\theta \, \{(n-1)\sin^{n-2}\theta \cos\theta \log(2\sin\theta) + \sin^{n-2}\theta\cos\theta\}\, d\theta \\ &= (n-1)(h_{n-2}-h_n) + \int_0^{\frac{1}{2}\pi} (\sin^{n-2}\theta - \sin^n\theta)\, d\theta. \end{aligned}$$

Thus
$$h_n = \frac{n-1}{n} h_{n-2} + \frac{1}{n^2} \frac{n-3,\, n-5 \ldots 1}{n-2,\, n-4 \ldots 2} \frac{\pi}{2}, \tag{25}$$

by which the integrals h_n can be calculated in turn. Thus

$$\begin{aligned} h_2 &= \pi/8, \\ h_4 &= \frac{3}{4} h_2 + \frac{1}{4^2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2} \frac{3\,.\,1}{4\,.\,2} \left(\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} \right), \\ h_6 &= \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2} \frac{\pi}{2} \left(\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} \right) + \frac{1}{6^2} \frac{3\,.\,1}{4\,.\,2} \frac{\pi}{2} \\ &= \frac{\pi}{2} \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2} \left(\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \frac{1}{5\,.\,6} \right). \end{aligned}$$

Similarly
$$h_8 = \frac{\pi}{2} \frac{7\,.\,5\,.\,3\,.\,1}{8\,.\,6\,.\,4\,.\,2} \left(\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \frac{1}{5\,.\,6} + \frac{1}{7\,.\,8} \right),$$
and so on.

It may be remarked that the series within brackets, being equal to

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots,$$

approaches ultimately the limit log 2. A tabulation of the earlier members of the series of integrals will be convenient :—

Table I.

$$\begin{aligned} 2h_0/\pi &= 0 \\ 2h_2/\pi &= 1/4 &&= 0{\cdot}25 \\ 2h_4/\pi &= 7/32 &&= 0{\cdot}21875 \\ 2h_6/\pi &= 37/192 &&= 0{\cdot}19271 \\ 2h_8/\pi &= 533/3072 &&= 0{\cdot}17350 \\ 2h_{10}/\pi &= 1627/10240 &&= 0{\cdot}15889 \\ 2h_{12}/\pi &= 18107/122880 &&= 0{\cdot}14736 \\ 2h_{14}/\pi &= \ldots\ldots\ldots\ldots\ldots\ldots &&= 0{\cdot}13798 \\ 2h_{16}/\pi &= \ldots\ldots\ldots\ldots\ldots\ldots &&= 0{\cdot}13018 \\ 2h_{18}/\pi &= \ldots\ldots\ldots\ldots\ldots\ldots &&= 0{\cdot}12356 \\ 2h_{20}/\pi &= \ldots\ldots\ldots\ldots\ldots\ldots &&= 0{\cdot}11784 \end{aligned}$$

The last four have been calculated in sequence by means of (25).

In (24) we may, of course, replace $\sin\theta$ by $\cos\theta$ throughout. If both $\sin\theta$ and $\cos\theta$ occur, as in

$$\int_0^{\frac{1}{2}\pi} \sin^n\theta \cos^m\theta \log(2\sin\theta)\, d\theta, \tag{26}$$

where n and m are even, we may express $\cos^m\theta$ by means of $\sin\theta$, and so reduce (26) to integrals of the form (24). The particular case where $m = n$ is worthy of notice. Here

$$\int_0^{\frac{1}{2}\pi} \sin^n\theta \cos^n\theta \log(2\sin\theta)\, d\theta = \int_0^{\frac{1}{2}\pi} \sin^n\theta \cos^n\theta \log(2\cos\theta)\, d\theta$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{\sin^n 2\theta}{2^n} \log(2\sin 2\theta)\, d\theta = \frac{h_n}{2^{n+1}}. \tag{27}$$

A comparison of the two treatments gives a relation between the integrals h. Thus, if $n = 4$,

$$h_4 - 2h_6 + h_8 = h_4/2^5.$$

We now proceed to the calculation of the left-hand member of (17) with $\Psi = (b^2-y^2)^{-\frac{1}{2}}$, or, as it may be written,

$$\int_{-b}^{+b} \frac{dy}{\sqrt{(b^2-y^2)}} \left[\left(\gamma + \log\frac{ikr}{2}\right) \mathrm{J}_0(kr) + \frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2.4^2}\mathrm{S}_2 + \frac{k^6r^6}{2^2.4^2.6^2}\mathrm{S}_3 - \ldots\right]. \tag{28}$$

The leading term has already been found to be

$$\pi\left(\gamma + \log\frac{ikb}{4}\right). \tag{29}$$

In (28) r is equal to $\pm(y-\eta)$. Taking, as before,

$$y = b\cos\theta, \qquad \eta = b\cos\alpha,$$

we have

$$\int_0^{\pi} d\theta \Bigg[\left\{\gamma + \log\frac{ikb}{4} + \log \pm 2(\cos\theta - \cos\alpha)\right\} \mathrm{J}_0\{kb(\cos\theta - \cos\alpha)\}$$

$$+ \frac{k^2b^2(\cos\theta-\cos\alpha)^2}{2^2} - \frac{k^4b^4(\cos\theta-\cos\alpha)^4}{2^2.4^2}\cdot\frac{3}{2} + \frac{k^6b^6(\cos\theta-\cos\alpha)^6}{2^2.4^2.6^2}\cdot\frac{11}{6} - \ldots\Bigg]. \tag{30}$$

As regards the terms which do not involve $\log(\cos\theta - \cos\alpha)$, we have to deal merely with

$$\int_0^{\pi} (\cos\theta - \cos\alpha)^n\, d\theta, \tag{31}$$

where n is an even integer, which, on expansion of the binomial and integration by a known formula, becomes

$$\pi\Bigg[\frac{n-1\,.\,n-3\,.\,n-5\ldots 1}{n\,.\,n-2\,.\,n-4\ldots 2} + \frac{n\,.\,n-1}{1\,.\,2}\,\frac{n-3\,.\,n-5\ldots 1}{n-2\,.\,n-4\ldots 2}\cos^2\alpha$$

$$+ \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}\,\frac{n-5\,.\,n-7\ldots 1}{n-4\,.\,n-6\ldots 2}\cos^4\alpha + \ldots + \cos^n\alpha\Bigg]. \tag{32}$$

Thus, if $n=2$, we get $\pi\left[\frac{1}{2}+\cos^2\alpha\right]$. If $n=4$,

$$\pi\left[\frac{3\,.\,1}{4\,.\,2}+\frac{4\,.\,3}{1\,.\,2}\,\frac{1}{2}\cos^2\alpha+\cos^4\alpha\right],\quad\text{and so on.}$$

The coefficient of (31), or (32), in (30) is

$$(-1)^{\frac{1}{2}n}\frac{k^n b^n}{2^2\,.\,4^2\ldots n^2}\left[\gamma+\log\frac{ikb}{4}-\mathrm{S}_{\frac{1}{2}n}\right]. \tag{33}$$

At the centre of the aperture where $\eta=0$, $\cos\alpha=0$, (32) reduces to its first term. At the edges where $\cos\alpha=\pm1$, we may obtain a simpler form directly from (31). Thus

$$(31)=\int_0^\pi(1\pm\cos\theta)^n\,d\theta=2^n\pi\frac{2n-1\,.\,2n-3\ldots1}{2n\,.\,2n-2\ldots2}=\pi\frac{2n-1\,.\,2n-3\ldots1}{n\,.\,n-1\,.\,n-2\ldots1}. \tag{34}$$

For example, if $n=6$,

$$(34)=\pi\frac{11\,.\,9\,.\,7\,.\,5\,.\,3\,.\,1}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}=\frac{231\,\pi}{16}.$$

We have also in (30) to consider (n even)

$$\begin{aligned}
&2^{-n}\int_0^\pi d\theta\,(\cos\theta-\cos\alpha)^n\log\{\pm2(\cos\theta-\cos\alpha)\}\\
&=\int_0^\alpha d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{4\sin\frac{\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2}\right\}\\
&\quad+\int_\alpha^\pi d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{4\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}\right\}\\
&=\int_0^\pi d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{2\sin\frac{\theta+\alpha}{2}\right\}\\
&\quad+\int_0^\alpha d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{2\sin\frac{\alpha-\theta}{2}\right\}\\
&\quad+\int_\alpha^\pi d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{2\sin\frac{\theta-\alpha}{2}\right\}\\
&=2\int_0^{\frac{1}{2}\pi+\frac{1}{2}\alpha}d\phi\sin^n\phi\,\sin^n(\phi-\alpha)\log(2\sin\phi)\\
&\quad+2\int_0^{\frac{1}{2}\pi-\frac{1}{2}\alpha}d\phi\sin^n\phi\,\sin^n(\phi+\alpha)\log(2\sin\phi)\\
&=2\int_0^{\frac{1}{2}\pi}d\phi\sin^n\phi\,\{\sin^n(\phi-\alpha)+\sin^n(\phi+\alpha)\}\log(2\sin\phi)\\
&\quad+2\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\frac{1}{2}\alpha}d\phi\sin^n\phi\,\sin^n(\phi-\alpha)\log(2\sin\phi)\\
&\quad-2\int_{\frac{1}{2}\pi-\frac{1}{2}\alpha}^{\frac{1}{2}\pi}d\phi\sin^n\phi\,\sin^n(\phi+\alpha)\log(2\sin\phi)\\
&=2\int_0^{\frac{1}{2}\pi}d\phi\sin^n\phi\,\{\sin^n(\phi-\alpha)+\sin^n(\phi+\alpha)\}\log(2\sin\phi),
\end{aligned} \tag{35}$$

since the last two integrals cancel, as appears when we write $\pi-\psi$ for ϕ, n being even.

In (35)

$$\tfrac{1}{2}\sin^n(\phi+\alpha)+\tfrac{1}{2}\sin^n(\phi-\alpha) = \sin^n\phi\cos^n\alpha$$

$$+\frac{n\,.\,n-1}{1\,.\,2}\sin^{n-2}\phi\cos^2\phi\sin^2\alpha\cos^{n-2}\alpha$$

$$+\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}\sin^{n-4}\phi\cos^4\phi\sin^4\alpha\cos^{n-4}\alpha+\ldots+\cos^n\phi\sin^n\alpha, \qquad (36)$$

and thus the result may be expressed by means of the integrals h. Thus if $n=2$,

$$(35) = 4\int_0^{\frac{1}{2}\pi} d\phi\sin^2\phi\,\{\sin^2\phi\cos^2\alpha+\cos^2\phi\sin^2\alpha\}\log(2\sin\phi)$$

$$= 4\,\{(\cos^2\alpha-\sin^2\alpha)\,h_4+\sin^2\alpha\,h_2\}. \qquad (37)$$

If $n=4$,

$$(35) = 4\int_0^{\frac{1}{2}\pi} d\phi\sin^4\phi\,\{\sin^4\phi\cos^4\alpha+6\sin^2\phi\cos^2\phi\sin^2\alpha\cos^2\alpha$$
$$+\cos^4\phi\sin^4\alpha\}\log(2\sin\phi)$$

$$= 4\,\{(\cos^4\alpha-6\sin^2\alpha\cos^2\alpha+\sin^4\alpha)\,h_8$$
$$+(6\sin^2\alpha\cos^2\alpha-2\sin^4\alpha)\,h_6+\sin^4\alpha\,h_4\}. \qquad (38)$$

If $n=6$,

$$(35) = 4\,\{(\cos^6\alpha-15\cos^4\alpha\sin^2\alpha+15\cos^2\alpha\sin^4\alpha-\sin^6\alpha)\,h_{12}$$
$$+(15\cos^4\alpha\sin^2\alpha-30\cos^2\alpha\sin^4\alpha+3\sin^6\alpha)\,h_{10}$$
$$+(15\cos^2\alpha\sin^4\alpha-3\sin^6\alpha)\,h_8+\sin^6\alpha\,h_6\}. \qquad (39)$$

It is worthy of remark that if we neglect the small differences between the h's in (39), it reduces to $4\cos^6\alpha\,h_{12}$, and similarly in other cases.

When n is much higher than 6, the general expressions corresponding to (37), (38), (39) become complicated. If, however, $\cos\alpha$ be either 0, or ± 1, (36) reduces to a single term, viz., $\cos^n\phi$ or $\sin^n\phi$. Thus at the centre ($\cos\alpha=0$) from either of its forms

$$(35) = 2^{-n}\,.\,2h_n. \qquad (40)$$

On the other hand, at the edges ($\cos\alpha=\pm 1$)

$$(35) = 4\int_0^{\frac{1}{2}\pi} d\phi\sin^{2n}\phi\log(2\sin\phi) = 4h_{2n}. \qquad (41)$$

In (30), the object of our quest, the integral (35) occurs with the coefficient

$$(-1)^{\frac{1}{2}n}\frac{2^n k^n b^n}{2^2\,.\,4^2\,.\,6^2\ldots n^2}. \qquad (42)$$

Thus, expanded in powers of kb, (28) or (30) becomes

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\Bigg[\left\{\gamma+\log\frac{ikb}{4}-1\right\}\left\{\tfrac{1}{2}+\cos^2\alpha\right\}\\
&\quad+\frac{2^3\,.\,2h_4}{\pi}(\cos^2\alpha-\sin^2\alpha)+\frac{2^3\,.\,2h_2}{\pi}\sin^2\alpha\Bigg]\\
&\quad+\frac{\pi k^4b^4}{2^2\,.\,4^2}\Bigg[\left\{\gamma+\log\frac{ikb}{4}-\frac{3}{2}\right\}\left\{\frac{3}{8}+3\cos^2\alpha+\cos^4\alpha\right\}\\
&\quad+\frac{2^5\,.\,2h_8}{\pi}(\cos^4\alpha-6\cos^2\alpha\sin^2\alpha+\sin^4\alpha)\\
&\quad+\frac{2^5\,.\,2h_6}{\pi}(6\cos^2\alpha\sin^2\alpha-2\sin^4\alpha)+\frac{2^5\,.\,2h_4}{\pi}\sin^4\alpha\Bigg]\\
&\quad-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\Bigg[\left\{\gamma+\log\frac{ikb}{4}-\frac{11}{6}\right\}\left\{\frac{5}{16}+\frac{45}{8}\cos^2\alpha+\frac{15}{2}\cos^4\alpha+\cos^6\alpha\right\}\\
&\quad+\frac{2^7\,.\,2h_{12}}{\pi}(\cos^6\alpha-15\cos^4\alpha\sin^2\alpha+15\cos^2\alpha\sin^4\alpha-\sin^6\alpha)\\
&\quad+\frac{2^7\,.\,2h_{10}}{\pi}(15\cos^4\alpha\sin^2\alpha-30\cos^2\alpha\sin^4\alpha+3\sin^6\alpha)\\
&\quad+\frac{2^7\,.\,2h_8}{\pi}(15\cos^2\alpha\sin^4\alpha-3\sin^6\alpha)+\frac{2^7\,.\,2h_6}{\pi}\sin^6\alpha\Bigg]+\ldots\ldots \qquad (43)
\end{aligned}$$

At the centre of the aperture ($\cos\alpha=0$), in virtue of (40), a simpler form is available. We have

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\left[\frac{1}{2}\left(\gamma+\log\frac{ikb}{4}-1\right)+\frac{2h_2}{\pi}\right]\\
&\quad+\frac{\pi k^4b^4}{2^2\,.\,4^2}\left[\frac{3\,.\,1}{4\,.\,2}\left(\gamma+\log\frac{ikb}{4}-\frac{3}{2}\right)+\frac{2h_4}{\pi}\right]\\
&\quad-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\left[\frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\left(\gamma+\log\frac{ikb}{4}-\frac{11}{6}\right)+\frac{2h_6}{\pi}\right]\\
&\quad+\frac{\pi k^8b^8}{2^2\,.\,4^2\,.\,6^2\,.\,8^2}\left[\frac{7\,.\,5\,.\,3\,.\,1}{8\,.\,6\,.\,4\,.\,2}\left(\gamma+\log\frac{ikb}{4}-\frac{25}{12}\right)+\frac{2h_8}{\pi}\right]-\ldots \qquad (44)
\end{aligned}$$

Similarly at the edges, by (34), (41), we have

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\left[\frac{3\,.\,1}{2\,.\,1}\left(\gamma+\log\frac{ikb}{4}-1\right)+2^3\frac{2h_4}{\pi}\right]\\
&\quad+\frac{\pi k^4b^4}{2^2\,.\,4^2}\left[\frac{7\,.\,5\,.\,3\,.\,1}{4\,.\,3\,.\,2\,.\,1}\left(\gamma+\log\frac{ikb}{4}-\frac{3}{2}\right)+2^5\frac{2h_8}{\pi}\right]\\
&\quad-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\left[\frac{11\,.\,9\,.\,7\,.\,5\,.\,3\,.\,1}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}\left(\gamma+\log\frac{ikb}{4}-\frac{11}{6}\right)+2^7\frac{2h_{12}}{\pi}\right]+\ldots \qquad (45)
\end{aligned}$$

For the general value of α, (43) is perhaps best expressed in terms of $\cos\alpha$, equal to η/b. With introduction of the values of h, we have

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\left[\left(\gamma+\log\frac{ikb}{4}\right)\left(\cos^2\alpha+\frac{1}{2}\right)+\frac{1}{2}\cos^2\alpha-\frac{1}{4}\right]\\
&\quad+\frac{\pi k^4 b}{2^2\,.\,4^2}\left[\left(\gamma+\log\frac{ikb}{4}\right)\left(\cos^4\alpha+3\cos^2\alpha+\frac{3}{8}\right)+\frac{7}{12}\cos^4\alpha-\frac{5}{4}\cos^2\alpha-\frac{11}{32}\right]\\
&\quad-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\left[\left(\gamma+\log\frac{ikb}{4}\right)\left(\cos^6\alpha+\frac{15}{2}\cos^4\alpha+\frac{45}{8}\cos^2\alpha+\frac{5}{16}\right)\right.\\
&\quad\left.+\frac{37}{60}\cos^6\alpha-\frac{23}{8}\cos^4\alpha-\frac{159}{32}\cos^2\alpha-\frac{73}{192}\right]+\dots\,. \qquad (46)
\end{aligned}$$

These expressions are the values of

$$\int_{-b}^{+b}\frac{\mathrm{D}\,(kr)\,dy}{\sqrt{(b^2-y^2)}} \qquad (47)$$

for the various values of η.

We now suppose that $kb=1$. The values for other particular cases, such as $kb=\frac{1}{2}$, may then easily be deduced. For $\cos\alpha=0$, from (44) we have

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{i}{4}\right)\left[1-\frac{1}{2^2}\,\frac{1}{2}+\frac{1}{2^2\,.\,4^2}\,\frac{3\,.\,1}{4\,.\,2}-\frac{1}{2^2\,.\,4^2\,.\,6^2}\,\frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}+\dots\right]\\
&\quad+\pi\left[\frac{1}{2^2}\,\frac{1}{4}-\frac{1}{2^2\,.\,4^2}\,\frac{11}{32}+\frac{1}{2^2\,.\,4^2\,.\,6^2}\,\frac{73}{192}-\dots\right]\\
&=\pi\left(\gamma+\log\frac{i}{4}\right)[1-0{\cdot}12500+0{\cdot}00586+0{\cdot}00013]\\
&\quad+\pi\,[0{\cdot}06250-0{\cdot}00537+0{\cdot}00016]\\
&=\pi\left(\gamma+\log\frac{i}{4}\right)\times 0{\cdot}88073+\pi\times 0{\cdot}05729\\
&=\pi\,[-0{\cdot}65528+1{\cdot}3834\,i], \qquad (48)
\end{aligned}$$

since $\gamma=0{\cdot}577215, \qquad \log 2=0{\cdot}693147, \qquad \log i=\frac{1}{2}\pi i.$

In like manner, if $kb=\frac{1}{2}$, we get still with $\cos\alpha=0$,

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{i}{8}\right)[1-0{\cdot}03125+0{\cdot}00037]+\pi\,[0{\cdot}01562-0{\cdot}00033]\\
&\qquad=\pi\,[-1{\cdot}4405+1{\cdot}5223\,i]. \qquad (49)
\end{aligned}$$

If $kb=2$, we have

$$\begin{aligned}
&\pi\left(\gamma+\log\frac{i}{2}\right)[1-0{\cdot}5+0{\cdot}0938-0{\cdot}0087+0{\cdot}0005]\\
&\quad+\pi\,[0{\cdot}25-0{\cdot}0859+0{\cdot}0102-0{\cdot}0006]\\
&=\pi\,[+0{\cdot}1058+0{\cdot}9199\,i]. \qquad (50)
\end{aligned}$$

If $kb = 1$ and $\cos\alpha = \pm 1$, we have from (45)

$$\pi\left(\gamma+\log\frac{i}{4}\right)\left[1-\frac{1}{2^2}\,\frac{3}{2}+\frac{1}{2^2.4^2}\,\frac{35}{8}-\frac{1}{2^2.4^2.6^2}\,\frac{231}{16}\right.$$
$$\left.+\frac{1}{2^2.4^2.6^2.8^2}\,\frac{6435}{128}-\frac{1}{2^2.4^2.6^2.8^2.10^2}\,\frac{19.17.6435}{10.9.128}+\ldots\right]$$
$$-\pi\left[\frac{1}{2^2}\,\frac{1}{4}+\frac{1}{2^2.4^2}\,\frac{97}{96}-\frac{1}{2^2.4^2.6^2}\,\frac{7303}{960}+\frac{38{\cdot}084}{2^2.4^2.6^2.8^2}-\frac{170{\cdot}64}{2^2.4^2.6^2.8^2.10^2}+\ldots\right]$$
$$=\pi\left(\gamma+\log\frac{i}{4}\right)[1-0{\cdot}375+0{\cdot}068359-0{\cdot}006266+0{\cdot}000341-0{\cdot}000012]$$
$$-\pi[0{\cdot}0625+0{\cdot}015788-0{\cdot}003302+0{\cdot}000258+0{\cdot}000012]$$
$$=\pi[-0{\cdot}63141+1{\cdot}0798\,i]. \qquad (51)$$

Similarly, if $kb = \frac{1}{2}$, we have

$$\pi\left(\gamma+\log\frac{i}{8}\right)[1-0{\cdot}09375+0{\cdot}00427-0{\cdot}00010]$$
$$-\pi[0{\cdot}01562+0{\cdot}00099-0{\cdot}00005]$$
$$=\pi[-1{\cdot}3842+1{\cdot}4301\,i]. \qquad (52)$$

And if $kb = 2$, with diminished accuracy,

$$\pi\left(\gamma+\log\frac{i}{2}\right)[1-1{\cdot}5+1{\cdot}094-0{\cdot}401+0{\cdot}087-0{\cdot}012+0{\cdot}001]$$
$$-\pi[0{\cdot}25+0{\cdot}253-0{\cdot}211+0{\cdot}066-0{\cdot}012+0{\cdot}001]$$
$$=\pi[-0{\cdot}378+0{\cdot}422\,i]. \qquad (53)$$

As an intermediate value of α we will select $\cos^2\alpha = \frac{1}{2}$. For $kb = 1$ from (46)

$$\pi\left(\gamma+\log\frac{i}{4}\right)[1-0{\cdot}25+0{\cdot}03320-0{\cdot}00222+\ldots]$$
$$+\pi[0-0{\cdot}01286+0{\cdot}001522+\ldots]$$
$$=\pi[-0{\cdot}6432+1{\cdot}2268\,i]. \qquad (54)$$

Also, when $kb = \frac{1}{2}$,

$$\pi[-1{\cdot}4123+1{\cdot}4759\,i]. \qquad (55)$$

When $kb = 2$, only a rough value is afforded by (46), viz.,

$$\pi[-0{\cdot}16+0{\cdot}61\,i]. \qquad (56)$$

The accompanying table exhibits the various numerical results, the factor π being omitted.

Table II.

	$kb = \frac{1}{2}$.	$kb = 1$.	$kb = 2$.
$\cos\alpha = 0$	$-1{\cdot}4405+1{\cdot}5223\,i$	$-0{\cdot}65528+1{\cdot}3834\,i$	$+0{\cdot}1058+0{\cdot}9199\,i$
$\cos^2\alpha = \frac{1}{2}$	$-1{\cdot}4123+1{\cdot}4759\,i$	$-0{\cdot}6432+1{\cdot}2268\,i$	$-0{\cdot}16+0{\cdot}61\,i$
$\cos^2\alpha = 1$	$-1{\cdot}3842+1{\cdot}4301\,i$	$-0{\cdot}63141+1{\cdot}0798\,i$	$-0{\cdot}378+0{\cdot}422\,i$

As we have seen already the tabulated quantity when kb is very small takes the form $\gamma + \log ikb/4$, or $\log kb - 0{\cdot}8091 + 1{\cdot}5708i$, whatever may be the value of α. In this case the condition (17) can be completely satisfied with $\Psi = \mathrm{A}(b^2 - y^2)^{-\frac{1}{2}}$, A being chosen suitably. When kb is finite, (17) can no longer be satisfied for all values of α. But when $kb = \frac{1}{2}$, or even when $kb = 1$, the tabulated number does not vary greatly with α and we may consider (17) to be approximately satisfied if we make in the first case

$$\pi(-1{\cdot}4123 + 1{\cdot}4759i)\,\mathrm{A} = -1, \tag{57}$$

and in the second,

$$\pi(-0{\cdot}6432 + 1{\cdot}2268i)\,\mathrm{A} = -1. \tag{58}$$

The value of ψ, applicable to a point at a distance directly in front of the aperture is then, as in (16),

$$\psi = -\pi\mathrm{A}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}. \tag{59}$$

In order to obtain a better approximation we require the aid of a second solution with a different form of Ψ. When this is introduced, as an addition to the first solution and again with an arbitrary constant multiplier, it will enable us to satisfy (17) for two distinct values of α, that is of η, and thus with tolerable accuracy over the whole range from $\cos\alpha = 0$ to $\cos\alpha = \pm 1$. Theoretically, of course, the process could be carried further so as to satisfy (17) for any number of assigned values of $\cos\alpha$.

As the second solution we will take simply $\Psi = 1$, so that the left-hand member of (17) is

$$\int_0^{b+\eta} \mathrm{D}(kr)\,dr + \int_0^{b-\eta} \mathrm{D}(kr)\,dr. \tag{60}$$

If we omit k, which may always be restored by consideration of homogeneity, we have

$$\begin{aligned}(60) = \left(\gamma + \log\frac{i}{2}\right)&\left[b + \eta - \frac{(b+\eta)^3}{2^2 . 3} + \frac{(b+\eta)^5}{2^2 . 4^2 . 5} - \dots\right] \\ &+ \frac{(b+\eta)^3}{2^2 . 3} - \frac{(b+\eta)^5}{2^2 . 4^2 . 5}\mathrm{S}_2 + \frac{(b+\eta)^7}{2^2 . 4^2 . 6^2 . 7}\mathrm{S}_3 - \dots \\ &+ (b+\eta)\{\log(b+\eta) - 1\} - \frac{(b+\eta)^3}{2^2 . 3}\left\{\log(b+\eta) - \frac{1}{3}\right\} \\ &+ \frac{(b+\eta)^5}{2^2 . 4^2 . 5}\left\{\log(b+\eta) - \frac{1}{5}\right\} - \dots\end{aligned}$$

+ the same expression with the sign of η changed.

The leading term in (60) is thus

$$2b(\gamma - 1 + \log\tfrac{1}{2}i) + (b+\eta)\log(b+\eta) + (b-\eta)\log(b-\eta). \tag{61}$$

At the centre of the aperture ($\eta = 0$)

$$(61) = 2b\,\{\gamma - 1 + \log \tfrac{1}{2} ib\},$$

and at the edges ($\eta = \pm b$),

$$(61) = 2b\,\{\gamma - 1 + \log ib\}.$$

It may be remarked that in (61), the real part varies with η, although the imaginary part is independent of that variable.

The complete expression (60) naturally assumes specially simple forms at the centre and edges of the aperture. Thus, when $\eta = 0$,

$$(60) \div 2b = \left(\gamma + \log \frac{ib}{2}\right)\left[1 - \frac{b^2}{2^2 . 3} + \frac{b^4}{2^2 . 4^2 . 5} - \ldots\right]$$
$$-1 + \frac{b^2}{2^2 . 3}\left(1 + \frac{1}{3}\right) - \frac{b^4}{2^2 . 4^2 . 5}\left(1 + \frac{1}{2} + \frac{1}{5}\right) + \frac{b^6}{2^2 . 4^2 . 6^2 . 7}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right) - \ldots; \qquad (62)$$

and, similarly, when $\eta = \pm b$,

$$(60) \div 2b = (\gamma + \log ib)\left[1 - \frac{(2b)^2}{2^2 . 3} + \frac{(2b)^4}{2^2 . 4^2 . 5} - \ldots\right]$$
$$-1 + \frac{(2b)^2}{2^2 . 3}\left(1 + \frac{1}{3}\right) - \frac{(2b)^4}{2^2 . 4^2 . 5}\left(1 + \frac{1}{2} + \frac{1}{5}\right) + \frac{(2b)^6}{2^2 . 4^2 . 6^2 . 7}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right) - \ldots . \qquad (63)$$

To restore k we have merely to write kb for b in the *right-hand members* of (62), (63).

The calculation is straightforward. For the same values as before of kb and of $\cos^2\alpha$, equal to η^2/b^2, we get for $(60) \div 2b$

Table III.

η^2/b^2.	$kb = \frac{1}{2}$.	$kb = 1$.	$kb = 2$.
0	$-1 \cdot 7649 + 1 \cdot 5384\, i$	$-1 \cdot 0007 + 1 \cdot 4447\, i$	$-0 \cdot 2167 + 1 \cdot 1198\, i$
$\frac{1}{2}$	$-1 \cdot 4510 + 1 \cdot 4912\, i$	$-0 \cdot 6740 + 1 \cdot 2771\, i$	$-0 \cdot 1079 + 0 \cdot 7166\, i$
1	$-1 \cdot 0007 + 1 \cdot 4447\, i$	$-0 \cdot 2217 + 1 \cdot 1198\, i$	$+0 \cdot 1394 + 0 \cdot 4024\, i$

We now proceed to combine the two solutions, so as to secure a better satisfaction of (17) over the width of the aperture. For this purpose we determine A and B in

$$\Psi = \mathrm{A}\,(b^2 - y^2)^{-\frac{1}{2}} + \mathrm{B}, \qquad (64)$$

so that (17) may be exactly satisfied at the centre and edges ($\eta = 0$, $\eta = \pm b$). The departure from (17) when $\eta^2/b^2 = \frac{1}{2}$ can then be found. If for any value of kb and $\eta = 0$ the first tabular (complex) number is p and the second q, and for $\eta = \pm b$ the first is r and the second s, the equations of condition from (17) are

$$\pi \mathrm{A} . p + 2b\mathrm{B} . q = -1, \qquad \pi \mathrm{A} . r + 2b\mathrm{B} . s = -1. \qquad (65)$$

When A and B are found, we have in (16)

$$\int_{-b}^{+b}\Psi\,dy = \pi\mathrm{A}+2b\mathrm{B}.$$

From (65) we get

$$\pi\mathrm{A} = \frac{q-s}{ps-qr}, \qquad 2b\mathrm{B} = \frac{r-p}{ps-qr}, \tag{66}$$

so that

$$\int_{-b}^{+b}\Psi\,dy = \frac{q+r-s-p}{ps-qr}. \tag{67}$$

Thus for $kb = 1$ we have

$$p = -0{\cdot}65528+1{\cdot}3834i, \qquad q = -1{\cdot}0007+1{\cdot}4447i,$$
$$r = -0{\cdot}63141+1{\cdot}0798i, \qquad s = -0{\cdot}2217+1{\cdot}1198i,$$

whence

$$\pi\mathrm{A} = +0{\cdot}60008+0{\cdot}51828i, \qquad 2b\mathrm{B} = -0{\cdot}2652+0{\cdot}1073i,$$

and

$$(67) = +0{\cdot}3349+0{\cdot}6256i.$$

The above values of πA and $2b$B are derived according to (17) from the values at the centre and edges of the aperture. The success of the method may be judged by substitution of the values for $\eta^2/b^2 = \frac{1}{2}$. Using these in (17) we get $-0{\cdot}9801-0{\cdot}0082\,i$, for what should be -1, a very fair approximation.

In like manner, for $kb = 2$

$$(67) = +0{\cdot}259+1{\cdot}2415\,i;$$

and for $kb = \frac{1}{2}$

$$(67) = +0{\cdot}3378+0{\cdot}3526\,i.$$

As appears from (16), when k is given, the modulus of (67) may be taken to represent the amplitude of disturbance at a distant point immediately in front, and it is this with which we are mainly concerned. The following table gives the values of Mod. and Mod.^{2} for several values of kb. The first three have been calculated from the simple formula, see (20).

Table IV.

kb.	Mod.^{2}.	Mod.
0·01	0·0174	0·1320
0·05	0·0590	0·2429
0·25	0·1372	0·3704
0·50	0·2384	0·4883
1·00	0·5035	0·7096
2·00	1·608	1·268

The results are applicable to the problem of aerial waves, or shallow water waves, transmitted through a slit in a thin fixed wall, and to electric (luminous) waves transmitted by a similar slit in a thin perfectly opaque screen, provided that the electric vector is *perpendicular* to the length of the slit.

In curve A, fig. 1, the value of the modulus from the third column of Table IV is plotted against kb.

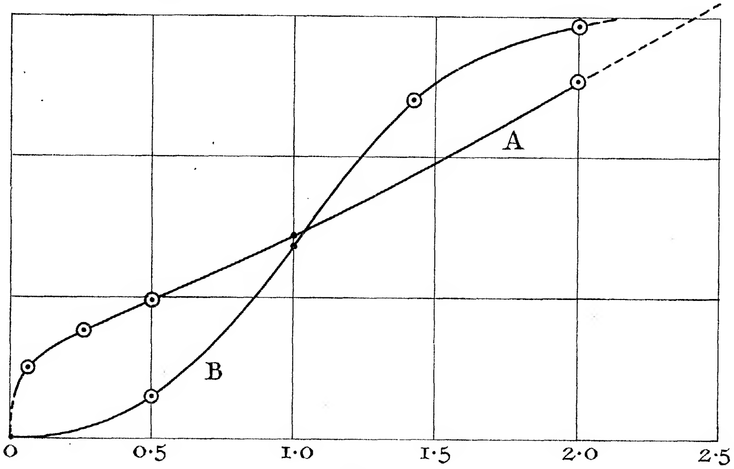

FIG. 1.

When kb is large, the limiting form of (67) may be deduced from a formula, analogous to (12), connecting Ψ and $d\phi/dn$. As in (11),

$$\frac{d\psi}{dx} = \int \Psi \frac{d\text{D}}{dx}\,dy,$$

in which, when x is very small, we may take $\text{D} = \log r$. Thus

$$\frac{d\psi}{dx} = \Psi \int_{-\infty}^{+\infty} \frac{x\,dy}{x^2+y^2} = \Psi\left[\tan^{-1}\frac{y}{x}\right]_{-\infty}^{+\infty} = \pi\Psi, \qquad \text{or} \qquad \Psi = \frac{1}{\pi}\frac{dx}{dn}. \quad (68)$$

Now, when kb is large, $d\psi/dn$ tends, except close to the edges, to assume the value ik, and ultimately

$$(67) = \int_{-b}^{+b} \Psi\,dy = \frac{2ikb}{\pi}, \qquad (69)$$

of which the modulus is $2kb/\pi$ simply, *i.e.* $0{\cdot}637kb$.

We now pass on to consider case (ii), where the boundary condition to be satisfied over the wall is $\phi = 0$. Separating from ϕ the solution (χ) which would obtain were the wall unperforated, we have

$$\chi_m = e^{-ikx} - e^{ikx}, \qquad \chi_p = 0, \qquad (70)$$

giving over the whole plane ($x = 0$),

$$\chi_m = 0, \qquad \chi_p = 0, \qquad d\chi_m/dx = -2ik, \qquad d\chi_p/dx = 0.$$

The supplementary solutions ψ, equal to $\phi-\chi$, may be written

$$\psi_m = \int \frac{d\mathrm{D}}{dx} \Psi_m \, dy, \qquad \psi_p = \int \frac{d\mathrm{D}}{dx} \Psi_p \, dy, \tag{71}$$

where Ψ_m, Ψ_p are functions of y, and the integrations are over the aperture. D as a function of r is given by (13), and r, denoting the distance between dy and the point (x, η), at which ψ_m, ψ_p are estimated, is equal to $\sqrt{\{x^2+(y-\eta)^2\}}$. The form (71) secures that on the *walls* $\psi_m = \psi_p = 0$, so that the condition of evanescence there, already satisfied by χ, is not disturbed. It remains to satisfy over the *aperture*.

$$\psi_m = \psi_p, \qquad -2ik + d\psi_m/dx = d\psi_p/dx. \tag{72}$$

The first of these is satisfied if $\Psi_m = -\Psi_p$, so that ψ_m and ψ_p are equal at any pair of corresponding points on the two sides. The values of $d\psi_m/dx$, $d\psi_p/dx$ are then opposite, and the remaining condition is also satisfied if

$$d\psi_m/dx = ik, \qquad d\psi_p/dx = -ik. \tag{73}$$

At a distance, and if the slit is very narrow, $d\mathrm{D}/dx$ may be removed from under the integral sign, so that

$$\psi_p = \frac{d\mathrm{D}}{dx} \int_{-b}^{+b} \Psi_p \, dy, \tag{74}$$

in which

$$\frac{d\mathrm{D}}{dx} = \frac{ikx}{r} \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}. \tag{75}$$

And, even if kb be not small, (74) remains applicable if the distant point be directly in front of the slit, so that $x = r$. For such a point

$$\psi_p = ik \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr} \int_{-b}^{+b} \Psi_p \, dy. \tag{76}$$

There is a simple relation, analogous to (68), between the value of Ψ_p at any point (η) of the aperture and that of ψ_p at the same point. For in the application of (71) only those elements of the integral contribute which lie infinitely near the point where ψ_p is to be estimated, and for these $d\mathrm{D}/dx = x/r^2$. The evaluation is effected by considering in the first instance a point for which x is finite and afterwards passing to the limit. Thus

$$\psi_p = \Psi_p \int \frac{x \, dy}{x^2+(y-\eta)^2} = \pi \Psi_p. \tag{77}$$

It remains to find, if possible, a form for Ψ_p, or ψ_p, which shall make $d\psi_p/dx$ constant over the aperture, as required by (73). In my former paper, dealing with the case where kb is very small, it was shown that known theorems relating to the flow of incompressible fluids lead to the desired conclusion. It appeared that (74), (75) give

$$\psi_p = -\frac{k^2b^2x}{2r}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}, \tag{78}$$

showing that when b is small the transmission falls off greatly, much more than in case (i), see (20). The realised solution from (78) is

$$\psi_p = -\frac{k^2b^2x}{2r}\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \cos(nt - kr - \tfrac{1}{4}\pi), \tag{79}$$

corresponding to $$\chi_m = 2\sin nt \sin kx. \tag{80}$$

The former method arrived at a result by assuming certain hydrodynamical theorems. For the present purpose we have to go further, and it will be appropriate actually to verify the constancy of $d\psi/dx$ over the aperture as resulting from the assumed form of Ψ, when kb is small. In this case we may take $\mathrm{D} = \log r$, where $r^2 = x^2 + (y-\eta)^2$. From (71), the suffix p being omitted,

$$\frac{d\psi}{dx} = \int_{-b}^{+b} \frac{d^2\mathrm{D}}{dx^2}\Psi\, dy;$$

and herein $$\frac{d^2\mathrm{D}}{dx^2} = -\frac{d^2\mathrm{D}}{d\eta^2} = -\frac{d^2\mathrm{D}}{dy^2} (\eta \text{ const.}).$$

Thus, on integration by parts,

$$\frac{d\psi}{dx} = -\left[\Psi\frac{d\mathrm{D}}{dy}\right] + \int_{-b}^{+b}\frac{d\mathrm{D}}{dy}\frac{d\Psi}{dy}\, dy. \tag{81}$$

In (81) $$\frac{d\mathrm{D}}{dy} = \frac{d\mathrm{D}}{dr}\frac{dr}{dy} = \frac{y-\eta}{(y-\eta)^2 + x^2},$$

and so long as η is not equal to $\pm b$, it does not become infinite at the limits ($y = \pm b$), even though $x = 0$. Thus, if Ψ vanish at the limits, the integrated terms in (81) disappear. We now assume for trial

$$\Psi = \sqrt{(b^2 - y^2)}, \tag{82}$$

which satisfies the last-mentioned condition. Writing

$$y = b\cos\theta, \qquad \eta = b\cos\alpha, \qquad x' = x/b,$$

we have $$-\frac{d\psi}{dx} = \int_0^{\pi} \frac{(\cos\theta - \cos\alpha)^2 + \cos\alpha\,(\cos\theta - \cos\alpha)}{(\cos\theta - \cos\alpha)^2 + x'^2}\, d\theta. \tag{83}$$

Of the two parts of the integral on the right in (83) the first yields π when $x' = 0$. For the second we have to consider

$$\int_0^{\pi} \frac{\cos\theta - \cos\alpha}{(\cos\theta - \cos\alpha)^2 + x'^2}\, d\theta, \tag{84}$$

in which $\cos\theta - \cos\alpha$ passes through zero within the range of integration. It will be shown that (84) vanishes ultimately when $x' = 0$. To this end the range of integration is divided into three parts: from 0 to α_1, where $\alpha_1 < \alpha$, from α_1 to α_2 where $\alpha_2 > \alpha$, and lastly from α_2 to π. In evaluating the first and third parts we may put $x' = 0$ at once. And if $z = \tan\frac{1}{2}\theta$

$$\int\frac{d\theta}{\cos\theta - \cos\alpha} = \frac{1}{\sin\alpha}\int\left\{\frac{dz}{\tan\frac{1}{2}\alpha + z} + \frac{dz}{\tan\frac{1}{2}\alpha - z}\right\}.$$

Sin α being omitted, the first and third parts together are thus

$$\log\frac{z+t}{z-t} + \log\frac{t+t_1}{t-t_1} + \log\frac{t_2-t}{t_2+t},$$

where $t = \tan\frac{1}{2}\alpha$, $t_1 = \tan\frac{1}{2}\alpha_1$, $t_2 = \tan\frac{1}{2}\alpha_2$, and z is to be made infinite.

It appears that the two parts taken together vanish, provided t_1, t_2 are so chosen that $t^2 = t_1 t_2$.

It remains to consider the second part, viz.,

$$\int_{\alpha_1}^{\alpha_2}\frac{d\theta\,(\cos\theta - \cos\alpha)}{(\cos\theta - \cos\alpha)^2 + x'^2}, \tag{85}$$

in which we may suppose the range of integration $\alpha_2 - \alpha_1$ to be very small. Thus

$$(85) = \int_{\alpha_1}^{\alpha_2}\frac{d\theta\,.\,2\sin\frac{1}{2}(\theta+\alpha)\ \sin\frac{1}{2}(\alpha-\theta)}{4\sin^2\frac{1}{2}(\theta+\alpha)\ \sin^2\frac{1}{2}(\alpha-\theta) + x'^2}$$

$$= -\frac{1}{2\sin\alpha}\log\frac{\sin^2\alpha\,(\alpha_2-\alpha)^2 + x'^2}{\sin^2\alpha\,(\alpha-\alpha_1)^2 + x'^2},$$

and this also vanishes if $\alpha_2 - \alpha = \alpha - \alpha_1$, a condition consistent with the former to the required approximation. We infer that in (83)

$$-\frac{d\Psi}{dx} = \pi, \tag{86}$$

so that, with the aid of a suitable multiplier, (73) can be satisfied. Thus if $\Psi = \mathrm{A}\sqrt{(b^2 - y^2)}$, (73) gives $\mathrm{A} = ik/\pi$, and the introduction of this into (74) gives (78). We have now to find what departure from (86) is entailed when kb is no longer very small.

Since, in general,

$$d^2\mathrm{D}/dx^2 + d^2\mathrm{D}/dy^2 + k^2\mathrm{D} = 0,$$

we find, as in (81),

$$-\frac{d\Psi}{dx} = k^2\int\Psi\,\mathrm{D}\,dy - \int\frac{d\Psi}{dy}\frac{d\mathrm{D}}{dy}dy, \tag{87}$$

and for the present Ψ has the value defined in (82). The first term on the right of (87) may be treated in the same way as (28) of the former problem, the difference being that $\sqrt{(b^2 - y^2)}$ occurs now in the numerator instead of the denominator. In (30) we are to introduce under the integral sign the additional factor $k^2b^2 \sin^2\theta$. As regards the second term of (87) we have

$$-\int \frac{d\Psi}{dy}\frac{d\mathrm{D}}{dy}\,dy = \int_{-b}^{+b} \frac{y\,(y-\eta)\,dy}{\sqrt{(b^2-y^2)}}\,\frac{1}{r}\frac{d\mathrm{D}}{dr},$$

where in $\frac{1}{r}\frac{d\mathrm{D}}{dr}$ we are to replace r by $\pm\,(y-\eta)$. We then assume as before $= b\cos\theta$, $\eta = b\cos\alpha$, and the same definite integrals h_n suffice; but the calculations are more complicated.

We have seen already that the leading term in (87) is π. For the next term we have

$$\mathrm{D} = \gamma + \log\frac{ikr}{2}, \qquad \frac{1}{r}\frac{d\mathrm{D}}{dr} = \frac{k^2}{4} - \frac{k^2}{2}\left(\gamma + \log\frac{ikr}{2}\right),$$

and thus

$$-\frac{1}{k^2b^2}\frac{d\psi}{dx} = \frac{\pi}{4}\left(\gamma + \log\frac{ikb}{4} + \frac{1}{2}\right)$$
$$+\int_0^\pi d\theta\,(1 - \tfrac{3}{2}\cos^2\theta + \tfrac{1}{2}\cos\alpha\cos\theta)\log\pm 2\,(\cos\theta - \cos\alpha). \quad (88)$$

The latter integral may be transformed into

$$2\int_0^{\frac{1}{2}\pi} d\phi\,\{1 - \tfrac{3}{2}\cos^2(2\phi - \alpha) + \tfrac{1}{2}\cos\alpha\cos(2\phi-\alpha)$$
$$+ 1 - \tfrac{3}{2}\cos^2(2\phi+\alpha) + \tfrac{1}{2}\cos\alpha\,(\cos 2\phi + \alpha)\}\log(2\sin\phi),$$

and this by means of the definite integrals h is found to be

$$-\frac{\pi}{8}(1 + 2\sin^2\alpha).$$

To this order of approximation the complete value is

$$-\frac{d\psi}{dx} = \pi + \tfrac{1}{4}\pi k^2b^2\,(\gamma - \sin^2\alpha + \log\tfrac{1}{4}ikb). \quad (89)$$

For the next two terms I find

$$+\frac{\pi k^4b^4}{512}[(1 + 4\cos^2\alpha)(1 - 4\gamma - 4\log\tfrac{1}{4}ikb)$$
$$+ 3\sin^4\alpha + \tfrac{1}{3}\cos^4\alpha + 6\sin^2\alpha\cos^2\alpha]$$
$$+\frac{\pi k^6b^6}{2^2.4^2.6}[(\tfrac{1}{16} + \tfrac{3}{4}\cos^2\alpha + \tfrac{1}{2}\cos^4\alpha)(\gamma + \log\tfrac{1}{4}ikb - \tfrac{2}{3})$$
$$+\frac{157}{8^2.15}\cos^6\alpha - \frac{13}{8^2.3}\cos^4\alpha\sin^2\alpha - \frac{15}{8^2}\cos^2\alpha\sin^4\alpha - \frac{7}{8^2.3}\sin^6\alpha]. \quad (90)$$

When $\cos\alpha = 0$, or ± 1, the calculation is simpler. Thus, when $\cos\alpha = 0$,

$$-\frac{1}{\pi}\frac{d\psi}{dx} = 1 + \frac{k^2b^2}{4}\left(\gamma + \log\frac{ikb}{4} - 1\right) - \frac{k^4b^4}{128}\left(\gamma + \log\frac{ikb}{4} - 1\right)$$
$$+\frac{k^6b^6}{6.4^5}\left(\gamma + \log\frac{ikb}{4} - \frac{5}{4}\right) - \frac{5\,k^8b^8}{9.4^9}\left(\gamma + \log\frac{ikb}{4} - \frac{22}{15}\right); \quad (91)$$

and when $\cos\alpha = \pm 1$,

$$-\frac{1}{\pi}\frac{d\psi}{dx} = 1+\frac{k^2b^2}{4}\left(\gamma+\log\frac{ikb}{4}\right)$$
$$-\frac{k^4b^4}{512}\left\{20\left(\gamma+\log\frac{ikb}{4}\right)-\frac{16}{3}\right\}+\frac{k^6b^6}{6\,.\,4^5}\left\{21\left(\gamma+\log\frac{ikb}{4}\right)-\frac{683}{60}\right\}$$
$$-\frac{k^8b^8}{9\,.\,4^9}\left\{429\left(\gamma+\log\frac{ikb}{4}\right)-329\right\}, \tag{92}$$

the last term, deduced from h_{14}, h_{16}, being approximate.

For the values of $-\pi^{-1}d\psi/dx$ we find from (91), (90), (92) for $kb = \frac{1}{2}, 1, \sqrt{2}, 2$:—

Table V.

	$kb = \frac{1}{2}$.	$kb = 1$.	$kb = \sqrt{2}$.	$kb = 2$.
$\cos\alpha = 0$	$0 \cdot 8448+0 \cdot 0974\,i$	$0 \cdot 5615+0 \cdot 3807\,i$	$0 \cdot 3123+0 \cdot 7383\,i$	$0 \cdot 0102+1 \cdot 3899\,i$
$\cos^2\alpha = \frac{1}{2}$	$0 \cdot 8778+0 \cdot 0958\,i$	$0 \cdot 6998+0 \cdot 3583\,i$	—	$0 \cdot 518\ +1 \cdot 129\,i$
$\cos^2\alpha = 1$	$0 \cdot 9103+0 \cdot 0944\,i$	$0 \cdot 8353+0 \cdot 3364\,i$	$0 \cdot 8587+0 \cdot 5783\,i$	$1 \cdot 020\ +0 \cdot 861\,i$

These numbers correspond to the value of Ψ expressed in (82).

We have now, in pursuance of our method, to seek a second solution with another form of Ψ. The first which suggests itself with $\Psi = 1$ does not answer the purpose. For (81) then gives as the leading term

$$-\frac{d\psi}{dx} = \left[\frac{y-\eta}{(y-\eta)^2+x^2}\right]_{-b}^{b} = \frac{2b}{b^2-\eta^2}, \tag{93}$$

becoming infinite when $\eta = \pm b$.

A like objection is encountered if $\Psi = b^2-y^2$. In this case

$$-\frac{d\psi}{dx} = 2\int\{(y-\eta)+\eta\}\frac{(y-\eta)\,dy}{(y-\eta)^2+x^2}.$$

The first part gives $4b$ simply when x becomes zero. And

$$2\int\frac{(y-\eta)\,dy}{(y-\eta)^2+x^2} = \log\frac{(b-\eta)^2+x^2}{(b+\eta)^2+x^2};$$

so that
$$-\frac{d\psi}{dx} = 4b+2\eta\log\frac{b-\eta}{b+\eta}, \tag{94}$$

becoming infinite when $\eta = \pm b$.

So far as this difficulty is concerned we might take $\Psi = (b^2-y^2)^2$, but another form seems preferable, that is

$$\Psi = b^{-2}(b^2-y^2)^{3/2}. \tag{95}$$

With the same notation as was employed in the treatment of (82) we have

$$-\frac{d\psi}{dx} = 3\int_0^\pi\frac{\cos\theta\,(\cos\theta-\cos\alpha)\,d\theta}{(\cos\theta-\cos\alpha)^2+x'^2}-3\int_0^\pi\frac{\cos^3\theta\,(\cos\theta-\cos\alpha)}{(\cos\theta-\cos\alpha)^2+x'^2}d\theta.$$

The first of these integrals is that already considered in (83). It yields 3π. In the second integral we replace $\cos^3\theta$ by $\{(\cos\theta-\cos\alpha)+\cos\alpha\}^3$, and we find, much as before, that when $x'=0$

$$\int_0^\pi \frac{\cos^3\theta\,(\cos\theta-\cos\alpha)\,d\theta}{(\cos\theta-\cos\alpha)^2+x'^2} = \pi\left(\tfrac{1}{2}+\cos^2\alpha\right). \tag{96}$$

Thus altogether for the leading term we get

$$-\frac{d\psi}{dx} = 3\pi\left(\tfrac{1}{2}-\cos^2\alpha\right) = 3\pi\left(\tfrac{1}{2}-\eta^2/b^2\right). \tag{97}$$

This is the complete solution for a fluid regarded as incompressible. We have now to pursue the approximation, using a more accurate value of D than that ($\log r$) hitherto employed.

In calculating the next term, we have the same values of D and $r^{-1}d\mathrm{D}/dr$ as for (88); and in place of that equation we now have

$$-\frac{1}{k^2b^2}\frac{d\psi}{dx} = \frac{3\pi}{16}\left(\gamma+\log\frac{ikb}{4}+\frac{1}{2}\right)$$
$$+\int_0^\pi d\theta\left[\tfrac{5}{2}\sin^4\theta-\tfrac{3}{2}\sin^2\theta+\tfrac{3}{2}\sin^2\theta\cos\theta\cos\alpha\right]\log\{\pm 2(\cos\theta-\cos\alpha)\}. \tag{98}$$

The integral may be transformed as before, and it becomes

$$\begin{aligned}4\int_0^{\frac{1}{2}\pi} d\phi\log(2\sin\phi)\Big[&\tfrac{5}{2}\left(\sin^4 2\phi\cos^4\alpha+6\sin^2 2\phi\cos^2 2\phi\sin^2\alpha\cos^2\alpha\right.\\ &\left.+\cos^4 2\phi\sin^4\alpha\right)-\tfrac{3}{2}\left(\sin^2 2\phi\cos^2\alpha+\cos^2 2\phi\sin^2\alpha\right)\\ &+\tfrac{3}{2}\cos\alpha\cos 2\phi\{\sin^2\alpha\cos\alpha+\sin^2 2\phi\,(\cos^3\alpha-3\sin^2\alpha\cos\alpha)\}\Big].\end{aligned} \tag{99}$$

The evaluation could be effected by expressing the square bracket in terms of powers of $\sin^2\phi$, but it may be much facilitated by use of two lemmas.

If $f(\sin 2\phi, \cos^2 2\phi)$ denote an integral function of $\sin 2\phi$, $\cos^2 2\phi$,

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} d\phi\log(2\sin\phi)f(\sin 2\phi,\cos^2 2\phi) &= \int_0^{\frac{1}{2}\pi} d\phi\log(2\cos\phi)f(\sin 2\phi,\cos^2 2\phi)\\ = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} d\phi\log(2\sin 2\phi)f(\sin 2\phi,\cos^2 2\phi) &= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} d\phi\log(2\sin\phi)f(\sin\phi,\cos^2\phi),\end{aligned} \tag{100}$$

in which the doubled angles are got rid of.

Again, if m be integral,

$$\begin{aligned}&\int_0^{\frac{1}{2}\pi} d\phi\sin^{2m}2\phi\cos 2\phi\log(2\sin\phi)\\ &= \frac{1}{4m+2}\int\log(2\sin\phi)\,d\sin^{2m+1}2\phi\\ &= -\frac{1}{4m+2}\int_0^{\frac{1}{2}\pi}\sin^{2m}2\phi\,(1+\cos 2\phi)\,d\phi\\ &= -\frac{1}{4m+2}\int_0^{\frac{1}{2}\pi}\sin^{2m}2\phi\,d\phi = -\frac{1}{4m+2}\int_0^{\frac{1}{2}\pi}\sin^{2m}\phi\,d\phi\\ &= -\frac{1}{4m+2}\,\frac{2m-1\,.\,2m-3\ldots 1}{2m\,.\,2m-2\ldots 2}\,\frac{\pi}{2}.\end{aligned} \tag{101}$$

For example, if $m = 0$,

$$\int_0^{\frac{1}{2}\pi} d\phi \cos 2\phi \log(2\sin\phi) = -\frac{\pi}{4}, \tag{102}$$

and ($m = 1$)

$$\int_0^{\frac{1}{2}\pi} d\phi \sin^2 2\phi \cos 2\phi \log(2\sin\phi) = -\frac{\pi}{24}. \tag{103}$$

Using these lemmas, we find

$$\begin{aligned}(99) = {} & 5h_4(\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha) \\ & + h_2(30\cos^2\alpha\sin^2\alpha - 10\sin^4\alpha - 3\cos^2\alpha + 3\sin^2\alpha) \\ & - \tfrac{1}{4}\pi\cos^2\alpha(\cos^2\alpha + 3\sin^2\alpha);\end{aligned}$$

and thence, on introduction of the values of h_2, h_4, for the complete value to this order of approximation,

$$\begin{aligned}-\frac{d\psi}{dx} = {} & 3\pi\left(\frac{1}{2} - \cos^2\alpha\right) + \pi k^2b^2\left[\frac{3}{16}\left(\gamma + \frac{1}{2} + \log\frac{ikb}{4}\right)\right. \\ & \left. - \frac{1}{64}(5\cos^4\alpha + 18\cos^2\alpha\sin^2\alpha + 21\sin^4\alpha)\right].\end{aligned} \tag{104}$$

To carry out the calculation to a sufficient approximation with the general value of α would be very tedious. I have limited myself to the extreme cases $\cos\alpha = 0$, $\cos\alpha = \pm 1$. For the former, we have

$$\begin{aligned}-\frac{1}{\pi}\frac{d\psi}{dx} = {} & \frac{3}{2} + \left(\gamma + \log\frac{ikb}{4}\right)\left\{\frac{3k^2b^2}{16} - \frac{k^4b^4}{256} + \frac{k^6b^6}{2^2.4^2.256}\right\} \\ & - \frac{15k^2b^2}{64} + \frac{7k^4b^4}{6.256} - \frac{11k^6b^6}{4^3.256.8},\end{aligned} \tag{105}$$

and for the latter

$$\begin{aligned}-\frac{1}{\pi}\frac{d\psi}{dx} = {} & -\frac{3}{2} + \left(\gamma + \log\frac{ikb}{4}\right)\left\{\frac{3k^2b^2}{16} - \frac{7k^4b^4}{16.16} + \frac{33k^6b^6}{4.16.16.16} - \frac{143k^8b^8}{24.16^4}\right\} \\ & - \frac{5k^2b^2}{64} + \frac{41k^4b^4}{16.64.15} + \frac{1069k^6b^6}{16.3.70.64.64} - \frac{41309k^8b^8}{16^5.9.420} \\ & + \frac{3k^2b^2}{32} + \frac{7k^4b^4}{4.16.16} - \frac{11k^6b^6}{2.16^3} + \frac{3289k^8b^8}{16^5.36}.\end{aligned} \tag{106}$$

From these formulæ the following numbers have been calculated for the value of $-\pi^{-1}d\psi/dx$:—

Table VI.

	$kb = \frac{1}{2}$.	$kb = 1$.	$kb = \sqrt{2}$.	$kb = 2$.
$\cos\alpha = 0$	$1{\cdot}3716 + 0{\cdot}0732\,i$	$1{\cdot}1215 + 0{\cdot}2885\,i$	$0{\cdot}8824 + 0{\cdot}5653\,i$	$0{\cdot}5499 + 1{\cdot}0860\,i$
$\cos\alpha = \pm 1$	$-1{\cdot}5634 + 0{\cdot}0710\,i$	$-1{\cdot}6072 + 0{\cdot}2546\,i$	$-1{\cdot}5693 + 0{\cdot}4401\,i$	$-1{\cdot}3952 + 0{\cdot}6567\,i$

They correspond to the value of Ψ formulated in (95).

Following the same method as in case (i), we now combine the two solutions, assuming

$$\Psi = A\sqrt{(b^2-y^2)}+Bb^{-2}(b^2-y^2)^{3/2}, \tag{107}$$

and determining A and B so that for $\cos\alpha = 0$ and for $\cos = \pm 1$, $d\psi/dx$ shall be equal to $-ik$. The value of ψ at a distance in front is given by (76), in which

$$ik\int \Psi\, dy = \frac{\pi\,.\,ikb^2}{2}\left(A+\frac{3}{4}B\right). \tag{108}$$

We may take the modulus of (108) as representing the transmitted vibrations, in the same way as the modulus of (67) represented the transmitted vibration in case (i).

Using p, q, r, s, as before, to denote the tabulated complex numbers, we have as the equations to determine A and B,

$$Ap+Bq = Ar+Bs = ik/\pi, \tag{109}$$

so that

$$ik\int \Psi\, dy = -\frac{k^2b^2}{2}\,\frac{s-q+\frac{3}{4}(p-r)}{ps-qr}. \tag{110}$$

For the second fraction on the right of (110) and for its modulus we get in the various cases

$kb = \frac{1}{2}$,	1·1470 − 0·1287*i*,	1·1542,
$kb = 1$,	1·1824 − 0·6986*i*,	1·3733,
$kb = \sqrt{2}$,	0·6362 − 1·0258*i*,	1·2070,
$kb = 2$,	0·1239 − 0·7303*i*,	0·7407.

And thence (on introduction of the value of kb) for the modulus of (110) representing the vibration on the same scale as in case (i).

Table VII.

kb.	Modulus.
$\frac{1}{2}$	0·1443
1	0·6866
$\sqrt{2}$	1·2070
2	1·4814

These are the numbers used in the plot of Curve B, fig. 1. When kb is much smaller than $\frac{1}{2}$, the modulus may be taken to be $\frac{1}{2}k^2b^2$. When kb is large the modulus approaches the same limiting form as in case (i).

This curve is applicable to electric, or luminous, vibrations incident upon a thin perfectly conducting screen with a linear perforation when the electric vector is *parallel* to the direction of the slit.

It appears that if the incident light be unpolarised, vibrations perpendicular to the slit preponderate in the transmitted light when the width of the slit is very small, and the more the smaller this width. In the neighbourhood of $kb = 1$, or $2b = \lambda/\pi$, the curves cross, signifying that the transmitted light is unpolarised. When $kb = 1\frac{1}{2}$, or $2b = 3\lambda/2\pi$, the polarisation is reversed, vibrations parallel to the slit having the advantage, but this advantage is not very great. When $kb > 2$, our calculations would hardly succeed, but there seems no reason for supposing that anything distinctive would occur. It follows that if the incident light were white and if the width of the slit were about one-third of the wave-length of yellow-green, there would be distinctly marked opposite polarisations at the ends of the spectrum.

These numbers are in good agreement with the estimates of Fizeau: "Une ligne polarisée perpendiculairement à sa direction a paru être de $\frac{1}{1000}$ de millimètre; une autre, beaucoup moins lumineuse, polarisée parallèlement a sa direction, a été estimée à $\frac{1}{10000}$ de millimètre. Je dois ajouter que ces valeurs ne sont qu'une approximation; elles peuvent être en réalité plus faibles encore, mais il est peu probable qu'elles soient plus fortes. Ce qu'il y a de certain, c'est que la polarisation parallèle n'apparaît que dans les fentes les plus fines, et alors que leur largeur est bien moindre que la longueur d'une ondulation qui est environ de $\frac{1}{2000}$ de millimètre." It will be remembered that the "plane of polarisation" is perpendicular to the electric vector.

It may be well to emphasize that the calculations of this paper relate to an aperture in an *infinitely thin* perfectly conducting screen. We could scarcely be sure beforehand that the conditions are sufficiently satisfied even by a scratch upon a silver deposit. The case of an ordinary spectroscope slit is quite different. It seems that here the polarisation observed with the finest practicable slits corresponds to that from the less fine scratches on silver deposits.